Power Series

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Start with a sequence $\{a_n\}_{n\geq 0}$ of real or complex numbers and set $s_n = \sum_{i=0}^{n} a_n$ to be the nth partial sum. We say that the series $\sum_{n\geq 0}$ (conditionally) converges to s if $\lim_{n\to\infty} s_n = s$. The series converges absolutely if $\sum_{n\geq 0} |a_n|$ converges to a finite real number. If the a_n are nonnegative reals, then observe that the sum as a limit of partial sums is the same as the supremum of all finite sums.

Lemma 1. *If a series converges absolutely, then it converges.*

Proof. Let $\{a_n\}_{n\geq 0}$ converge absolutely. It suffices to show that the sequence of partial sums is a Cauchy sequence, because both $\mathbb{R}, \mathbb{C}$ are complete metric spaces. Furthermore, because of absolute convergence, the sequence of partial sums $t_n = \sum_{i=0}^{n} |a_i|$ is Cauchy. Now given $\epsilon > 0$, choose N such that for $m, n \geq N, |t_m - t_n| < \epsilon$. Then for the same $N, m, n \geq N$ we have $|s_m - s_n| = |a_n + a_{n+1} + \cdots + a_m| \leq t_m - t_n < \epsilon$ assuming, without loss of generality, $m > n$. Therefore, the sequence $\{s_n\}_{n\geq 0}$ is Cauchy, hence $\sum_{n\geq 0} a_n$ converges. $\square$

Lemma 2. *If $\{c_n\}_{n\geq 0}$ is a sequence of positive real numbers and $\{a_n\}_{n\geq 0}$ is a sequence of complex numbers such that for each $n, |a_n| \leq c_n$, then*

1. *If the partial sums of $\{c_n\}_{n\geq 0}$ are bounded, then $\sum c_n$ converges.*
2. *If $\sum c_n$ converges, then $\{a_n\}_{n\geq 0}$ converges absolutely.*

Proof. The partial sums of $\{c_n\}_{n\geq 0}$ form a monotonically increasing sequence, hence converge to their supremum (which may be ∞). Hence $\sum c_n$ converges (absolutely) if and only if the partial sums are bounded. If $\sum c_n$ converges, then $\sum_{i=0}^{n} |a_n| \leq \sum_{i=0}^{n} c_n$, therefore $\{a_n\}_{n\geq 0}$ converges absolutely. $\square$

Lemma 3. *(Rearrangement theorem) Suppose $\{a_n\}_{n\geq 0}$ is absolutely convergent, then any permutation of the sequence is also absolutely convergent with the same sum. In other words, given a bijection $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, the sequence $\{b_n\}_{n\geq 0}$ with $b_n = a_{f(n)}$ is absolutely convergent with the same sum.*

Proof. First, given any n, we have $\sum_{i=0}^{n} |b_i| = \sum_{i=0}^{n} |a_{f(i)}| \leq \sum |a_n|$, therefore the partial sums of $|b_n|$ are increasing and bounded above, hence convergent. Therefore $\{b_n\}_{n\geq 0}$ is absolutely convergent.

Next, suppose $\sum a_n = a$, then given $\epsilon > 0$, there is an N such that for $n \geq N, |\sum_{i=0}^{n} a_i - a| < \epsilon$. Suppose also that N is chosen so that for $m \geq n \geq N$, we have $\sum_{i=n}^{m} a_i < \epsilon$. Now, pick N' such that $\{1, \ldots, N\} \subseteq \{f(1), \ldots, f(N')\}$. Then, for $m \geq N'$, we have

$$\begin{aligned} |\sum_{i=0}^{m} b_i - a| &\leq |\sum_{i=0}^{n} a_i - a| + |\sum_{i:f(i)\geq N} a_{f(i)}| \\ &< \epsilon + \epsilon \end{aligned}$$

where the second term is bounded by ϵ by the Cauchy criterion. Therefore, $\sum b_n = a$. A similar proof shows that $\sum |b_n| = \sum |a_n|$. $\square$

Remark. When the series is conditionally convergent but not absolutely, the Riemann rearrangement theorem [4] says that the sequence may be permuted so as to converge to any finite real number, diverge to $\pm\infty$ or not have a limit at all.

Lemma 4. *(Arithmetic progressions) Let $\{a_n\}_{n\geq 0}$ be absolutely convergent and suppose we partition $\mathbb{Z}_{\geq 0}$ into finitely many disjoint arithmetic progressions $P_1, \ldots, P_k$, then $\sum a_n = \sum_{a\in P_1} a + \cdots + \sum_{a\in P_k} a$.*

Proof. Because $\{a_n\}_{n\geq 0}$ converges absolutely, any subsequence also converges absolutely, so each term on the right is defined. Next, each partial sum of a_n can be grouped into partial sums from each progression. The result then follows. □

Theorem 1. *(Fubini-Tonelli theorem for sums) Suppose $a_{m,n}, m, n \geq 1$ is a coutable collection of real or complex numbers such that the sum (defined as the supremum of finite sums) $\sum_{m,n} |a_{m,n}| < \infty$, then both $\sum_m \sum_n a_{m,n}$ and $\sum_n \sum_m a_{m,n}$ are absolutely convergent and equal.*

Proof. Here the double sum $\sum_{m,n} |a_{m,n}|$ is defined as the supremum of all finite sums. Given that it is finite, we see that if m is fixed, then $\sum_n a_{m,n}$ converges absolutely, and similarly if n is fixed. Let A be the sum $\sum_{m,n} |a_{m,n}|$. We first show that $\sum_m \sum_n |a_{m,n}| = A$. For any M,

$$\sum_{m=1}^{M} \sum_{n} |a_{m,n}| = \sum_{m=1}^{M} \lim_{N\to\infty} \sum_{n=1}^{N} a_{m,n} \leq A.$$

Therefore, all partial sums $\sum_{m=1}^{M} \sum_n |a_{m,n}|$ are less than A, hence $\sum_m \sum_n |a_{m,n}|$ is finite and bounded by A.

Conversely, given $\epsilon > 0$, choose a finite subset F of $\mathbb{N}^2$ such that $A - \epsilon < \sum_{(m,n)\in F} |a_{m,n}| < A$, and let M be the largest first coordinates of points in F and N the largest second coordinates. Then,

$$A - \epsilon < \sum_{(m,n)\in F} |a_{m,n}| \leq \sum_{m=1}^{M} \sum_{n=1}^{N} |a_{m,n}| \leq \sum_{m=1}^{M} \sum_{n} |a_{m,n}| \leq A.$$

Therefore, $\sum_m \sum_n |a_{m,n}| = A$. Similarly, $\sum_n \sum_m |a_{m,n}| = A$. It follows that $\sum_m (\sum_n a_{m,n})$ is absolutely convergent for $\sum_m |\sum_n a_{m,n}| \leq \sum_m \sum_n |a_{m,n}| < \infty$. Similarly, the other iterated sum is absolutely convergent. Next, we show that both iterated sums are equal.

Set $S = \sum_m \sum_n a_{m,n} \in \mathbb{C}, T = \sum_n \sum_m a_{m,n} \in \mathbb{C}$. Suppose $S \neq T$, then because $\mathbb{C}$ is Hausdorff, there is a $2\epsilon > 0$ such that the open balls of radius 2ϵ around S, T are disjoint. Now, choose M, N such that

$$\sum_{m>M} \sum_{n} |a_{m,n}| < \epsilon, \sum_{n>N} \sum_{m} |a_{m,n}| < \epsilon,$$

(they exist because of absolute convergence) then

$$\begin{aligned}
|S - \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}| &\leq |S - \sum_{m=1}^{M} \sum_{n} a_{m,n}| + |\sum_{m=1}^{M} \sum_{n} a_{m,n} - \sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}| \\
&= |\sum_{m>M} \sum_{n} a_{m,n}| + |\sum_{m=1}^{M} \sum_{n>N} a_{m,n}| \\
&\leq \sum_{m>M} \sum_{n} |a_{m,n}| + \sum_{m=1}^{M} \sum_{n>N} |a_{m,n}| \\
&\leq 2\epsilon
\end{aligned}$$

where in the second term of the last step we have interchanged a limit with a finite sum, and interchanged a double finite sum. Therefore, $\sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}$ is within 2ϵ of S. Since we can interchange finite double sums, the exact same argument shows that $\sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n}$ is within 2ϵ of T. This is a contradiction, therefore we must have $S = T$. □

1 limsup and liminf

In this section we only consider sequences of real numbers. Moreover we deal with the extended reals $[-\infty, \infty]$ with the role of $\pm\infty$ only used for inequalities. Given $\{a_n\}_{n\geq 0}$, set

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{i\geq n} a_i$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{i\geq n} a_i$$

Because the sup, inf sequences are monotonic, a limit always exists (but may be $\pm\infty$). Moreover, $\limsup(\liminf)$ is finite if and only if $\{a_n\}_{n\geq 0}$ is bounded, i.e., bounded above and below.

Given two sequence $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$, we have for every $n, j \geq n$,

$$\inf_{i\geq n} a_i + \inf_{i\geq n} b_i \leq a_j + b_j \leq \sup_{i\geq n} a_i + \sup_{i\geq n} b_i.$$

Next, given $0 < c < \infty$, we have for every $n, j \geq n$, $c \inf_{i\geq n} a_i \leq ca_j \leq c \sup_{i\geq n} a_i$. The inequalities are reversed if $-\infty < c < 0$. Combining all these we get (and using $1/c$ to get the other inequality)

$$\sup_{i\geq n}(a_i + b_i) \leq \sup_{i\geq n} a_i + \sup_{i\geq n} b_i$$
$$\inf_{i\geq n}(a_i + b_i) \geq \inf_{i\geq n} a_i + \inf_{i\geq n} b_i$$

If $0 < c < \infty$,

$$\sup_{i\geq n} ca_i = c \sup_{i\geq n} a_i$$
$$\inf_{i\geq n} ca_i = c \inf_{i\geq n} a_i$$

If $-\infty < c < 0$,

$$\sup_{i\geq n} ca_i = c \inf_{i\geq n} a_i$$
$$\inf_{i\geq n} ca_i = c \sup_{i\geq n} a_i$$

Next, if both $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$ are positive, then for every n

$$\inf_{i\geq n} a_i \inf_{i\geq n} b_i \leq \inf_{i\geq n} a_i b_i \leq \sup_{i\geq n} a_i b_i \leq \sup_{i\geq n} a_i \sup_{i\geq n} b_i.$$

If $a_n \leq b_n \,\forall\, n$, then it is quite obvious that given $n, i \geq n, a_i \leq b_i \leq \sup_{i\geq n} b_i$, therefore $\sup_{i\geq n} a_i \leq \sup_{i\geq n} b_i$. Similarly, $\inf_{i\geq n} a_i \leq \inf_{i\geq n} b_i$.

All of the inequalities above translate to similar inequalities of $\limsup$ and $\liminf$. Lastly, we look at the question of convergence of $\{a_n\}_{n\geq 0}$.

Lemma 5. *$\{a_n\}_{n\geq 0}$ is convergent if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. In this case, the limit is equal to the limsup.*

Proof. It is obvious that for any $n, \inf_{i\geq n} a_n \leq \sup_{i\geq n} a_n$. Therefore,

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Suppose $a_n \to a$, then given $\epsilon > 0$, find N such that for $n \geq N, a_n \in (a - \epsilon, a + \epsilon)$. This means that

$$a - \epsilon \leq \inf_{i\geq n} a_i \leq \sup_{i\geq n} a_i \leq a_\epsilon.$$

Taking limits (which exist because a_n is bounded) we obtain

$$a - \epsilon \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq a + \epsilon.$$

Since $\epsilon > 0$ was arbitrary it follows that

$$\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n.$$

Conversely, suppose $\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$. Given $\epsilon > 0$, choose N such that for $n \geq N$,

$$a - \epsilon \leq \inf_{i\geq n} a_i \leq a \leq \sup_{i\geq n} a_i \leq a + \epsilon.$$

It follows that for $n \geq N, a_n \in (a-\epsilon, a+\epsilon)$, therefore $\{a_n\}_{n\geq 0}$ converges to a. $\square$

2 Tests for absolute convergence

Given a sequence $\{a_n\}_{n\geq 0}$, to show that it converges it suffices to show that any subsequence $\{a_n\}_{n\geq N}$ converges because the initial few terms are finite.

Theorem 2. *(Limit comparision test) Let $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0}$ be two strictly positive sequences of real numbers such that $r = \lim(x_n/y_n)$ exists, then*

- *If $r \neq 0$, then $\sum x_n$ converges if and only if $\sum y_n$ converges.*
- *If $r = 0$ and $\sum y_n$ converges, then $\sum x_n$ converges.*

Proof. Suppose $r \neq 0$, then choose $0 < \epsilon < r$, then there is an N such that for $n \geq N$, we have

$$(r-\epsilon)y_n < x_n < (r+\epsilon)y_n.$$

It follows, by comparision (both $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0}$ are positive), that $\sum y_n$ converges if and only if $\sum x_n$ converges. Suppose $r = 0$, then find an N such that for $n \geq N, x_n/y_n < 1$. Again by comparision, if $\sum y_n$ converges, then so does $\sum x_n$. $\square$

Theorem 3. *(Root test) Let $\{a_n\}_{n\geq 0}$ be a sequence and set $r = \limsup_{n\to\infty} |a_n|^{1/n}$. Then $\sum a_n$ converges if $r < 1$ and diverges if $r > 1$.*

Proof. First take $r < 1$ and pick $\epsilon > 0$ such that $r + \epsilon < 1$. Next choose N such that for $n \geq N, \sup_{i\geq n} |a_i|^{1/i} \leq r + \epsilon$. We then have for $n \geq N, |a_n| \leq (r+\epsilon)^n$. Since the right side is a geoemetric series with $r + \epsilon < 1$, it converges and by comparision $\{a_n\}_{n\geq 0}$ is absolutely converges.

Next, suppose $r > 1$, then choose $\epsilon > 0$ such that $1 < r - \epsilon$. Pick N such that for $n \geq N, \sup_{i\geq n} |a_i|^{1/i} \in (r-\epsilon, r+\epsilon)$. We must have infinitely many $|a_i|^{1/i}$ in this interval, so there are infinitely many terms such that $|a_i| > (r-\epsilon)^i > 1$. Therefore, $\sum |a_n|$ diverges. $\square$

Remark. Note that for continuity reasons $x^{1/n}$ always exists for positive x. This notion requires only the knowledge of how to take integer exponents. On the other hand, arbitrary exponentiation is a limiting operation.

Theorem 4. *(Ratio test) Let $\{a_n\}_{n\geq 0}$ be a sequence, set*

$$R = \limsup_{n\to\infty} |a_{n+1}/a_n| \text{ and } r = \liminf_{n\to\infty} |a_{n+1}/a_n|.$$

If $R < 1$, then $\{a_n\}_{n\geq 0}$ converges absolutely. If $r > 1$, then $\sum |a_n|$ diverges.

Proof. If $R < 1$, then choose ϵ such that $R + \epsilon < 1$, then there is an N such that for $n \geq N, |a_n| < (R+\epsilon)^{n-N}|a_N|$ and since $R + \epsilon < 1, \sum |a_n|$ converges. If $r > 1$¡ then take ϵ so that $r - \epsilon > 1$, and there is an N beyond which $|a_n| > (r-\epsilon)^{n-N}|a_N|$, therefore $\sum |a_n|$ diverges. $\square$

Remark. In both ratio and root tests, we cannot say what happens when the limsup or liminf is 1.

Theorem 5. *(Integral test) Let f be a positive decreasing integrable function on $[1,\infty)$. Then the series $\sum f(n)$ converges if and only if $\int_1^\infty f(t)dt\colon = \lim_{b\to\infty}\int_1^b f(t)dt$ exists. In this case, let the limit be s and partial sums s_n, then*

$$\int_{n+1}^\infty f(t)dt \le s - s_n \le \int_n^\infty f(t)dt.$$

Proof. Since f is positive and decreasing, $f(k) \le \int_{k-1}^k f(t)dt \le f(k-1)$. Adding the inequalities, we get $s_k - f(1) \le \int_1^k f(t)dt \le s_{k-1}$. Therefore, the partial sums converge if and only if the integral converges and in this case, if we sum the original inequality upwards, $s - s_{k-1} \le \int_{k-1}^\infty f(t)dt \le s - s_{k-2}$ from which the other inequality follows. □

Theorem 6. *(Raabe's test) Let $\{x_n\}_{n\ge 0}$ be a sequence.*

(a) If $\exists a > 1, K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \le 1 - \frac{a}{n} \forall n \ge K$$

then $\sum |x_n|$ converges.

(b) If $\exists a \le 1, K \in \mathbb{N}$ such that

$$\left|\frac{x_{n+1}}{x_n}\right| \ge 1 - \frac{a}{n} \forall n \ge K$$

then $\sum |x_n|$ doesn't converge.

Proof. Let K, a be given.

(a) For $n \ge K$, we have

$$0 < (a-1)|x_n| \le (n-1)|x_n| - n|x_n|.$$

Add the inequlities from K to $k \ge K$, then

$$(a-1)(|x_K| + \cdots + |x_k| \le (K-1)|x_K| - k|x_{k+1}| < (K-1)|x_K|.$$

Therefore the partial sums are bounded (because $a > 1$), hence $\{x_n\}_{n\ge 0}$ is absolutely convergent.

(b) We can take $K > 2$, then for every $n \ge K$, we have

$$n|x_{n+1}| \ge (n-a)|x_n| \ge (n-1)|x_n|.$$

Repeatedly applying this inequality, we have for every $n \ge K$,

$$|x_{n+1}| \ge \frac{(K-1)|x_K|}{n}.$$

Since $\sum 1/n$ diverges, so does $\sum |x_n|$. □

Corollary 1. *Given a sequence $\{x_n\}_{n\ge 0}$, suppose $a = \lim n(1 - |x_{n+1}/x_n|)$ exists. Then $\sum |x_n|$ converges when $a > 1$ and diverges when $a < 1$.*

3 Tests for non absolute convergence

We give certain conditions when a sequence may converge conditionally without converging absolutely.

Theorem 7. *(Alternating series) Let $\{z_n\}_{n\geq 0}$ be a decreasing sequence of positive numbers decreasing to zero, then $\sum(-1)^n z_n$ converges.*

Proof. Let s_n be the nth partial sum, then $s_{2n+1} = (z_0 - z_1) + (z_2 - z_3) + \cdots + (z_{2n} - z_{2n+1})$ is a sum of positive terms and the sequence $\{s_{2n+1}\}_{n\geq 0}$ is increasing. Furthermore, each s_{2n+1} is bounded above by s_0, therefore s_{2n+1} converges to some s. Now given $\epsilon > 0$ pick N such that for $n \geq N$, we have $z_n < \epsilon$ and $|s - s_{2n+1}| < \epsilon$. Then for any $k \geq 2N + 2$ we have $|s_k - s| < \epsilon$ if k is odd and if $k = 2t + 2$ for $t \geq N$, then $|s_k - s| \leq |s_{2t+1} - s| + |z_{2t+2}|$. In both cases, we see that the partial sums converge to s. □

Lemma 6. *(Abel's lemma) Let $\{x_n\}_{n\geq 0}, \{y_n\}_{n\geq 0}$ be two sequences in $\mathbb{C}$ and denote the partial sums of $\{y_n\}_{n\geq 0}$ by $\{s_n\}_{n\geq 0}$. Then for $m > n$,*

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Proof. For each $k \geq 1$, we have $y_k = s_k - s_{k-1}$. If we now collect the terms we get $\sum_{k=n+1}^{m} x_k(s_k - s_{k-1}) = x_m s_m - x_{n+1} s_n + \sum_{k=n+1}^{m-1} (x_k - x_{k-1}) s_k$. □

Theorem 8. *(Dirichlet's test) If $\{x_n\}_{n\geq 0}$ is decreasing to zero and if the partial sums of $\{y_n\}_{n\geq 0}$ are bounded then $\sum x_n y_n$ is convergent.*

Proof. Let the partial sums of $\{y_n\}_{n\geq 0}$ be bounded by $B > 0$, then given n, for $m > n$ using $x_k - x_{k+1} \geq 0$,

$$\begin{aligned} |\sum_{k=n+1}^{m} x_k y_k &\leq (x_m + x_{m+1})B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1})B \\ &= [(x_m + x_{n+1}) + (x_{n+1} - x_m)]B \\ &= 2x_{n+1}B \end{aligned}$$

Therefore, the partial sums form a Cauchy sequence, hence $\sum x_n y_n$ is convergent. □

Theorem 9. *(Abel's test) If $\{z_n\}_{n\geq 0}$ is a convergent monotone sequence and the series $\sum y_n$ is convergent, then $\sum x_n y_n$ is convergent.*

Proof. If x_n decreases to x, then take $u_n = x_n - x$ and u_n decreases to 0. Then by Dirichlet's test, $\sum u_n y_n$ converges therefore, $\sum x_n y_n = \sum x y_n + \sum u_n y_n$ converges. If x_n increases to x, then use the same method, but take $u_n = x - x_n$. Note that we can separate the sums into two because we are taking a limit of the partial sums, which separate into two, in the given ordering. □

4 Sequences of functions

Functions that are given by power series are basically the limits of the functions determined by the partial sums. Therefore, we first study sequences of functions and their various limits. Let $A \subseteq \mathbb{R}$, and for each $n \in \mathbb{Z}_{\geq 0}$, suppose we have a function $f_n \colon A \to \mathbb{R}$.

Definition 1. *Let $A \subseteq \mathbb{R}^m$ and $f_n \colon A \to \mathbb{R}^k$ be a sequence of functions, i.e., there is one function f_n for each $n \in \mathbb{Z}_{\geq 0}$. Let $A_0 \subseteq A$ and let $f \colon A_0 \to \mathbb{R}^k$. We say that the sequence $\{f_n\}_{n\geq 0}$ converges pointwise on A_0 to f if for every $x \in A_0, f_n(x) \to f(x)$.*

The convergence is uniform on A_0 if for every $\epsilon > 0$, there is an $N \geq 1$ such that for each $n \geq N, x \in A_0, |f_n(x) - f(x)| < \epsilon$.

Given a function $\phi \colon A \to \mathbb{R}^k$, it is said to be bounded if its image is bounded and in this case, we define the sup norm (or uniform norm) by $\|\phi\|_A = \sup\{|\phi(x)| : x \in A\}$.

Remark. More generally, we can replace A with any topological space and define the uniform norm similarly. The name norm is valid because one can check quite easily that $\|\|_A$ satisfies all the axioms of a norm on a vector space (that of bounded functions).

Observe that f_n are bounded and converge uniformly to f, then f is bounded for $|f(x)| \leq |f(x) - f_n(x)| + |f_n(x)|$.

Theorem 10. *(Cauchy criterion) Let $\{f_n\}_{n\geq 0}$ be a sequence of bounded functions on $A \subset \mathbb{R}^m$. This sequence converges uniformly on A to a bounded function f if and only if $\forall \epsilon > 0, \exists N$ such that for $m, n \geq N, \|f_m - f_n\|_A < \epsilon$.*

Proof. Suppose $\{f_n\}_{n\geq 0}$ converges uniformly to f, then we have $\|f_m - f_n\| \leq \|f_n - f\| + \|f_m - f\|$. Now, N can be chosen suitable so that the terms on the right are together less than ϵ.

Conversely, suppose $\{f_n\}_{n\geq 0}$ satisfies the Cauchy criterion, then for each $x \in A$, the sequence $\{f_n(x)\}$ is Cauchy, hence converges to some $f(x)$. Given any m, n we have

$$|f_m(x) - f(x)| \leq |f_m(x) - f_n(x)| + |f_n(x) - f(x)| \leq \|f_m - f_n\| + |f_n(x) - f(x)|.$$

Given $\epsilon > 0$, let N be chosen as stated and let $m, n \geq N$. Since the inequality holds for every n, it holds when we take the limit $n \to \infty$. Therefore, we get $\forall x \in A, |f_m(x) - f(x)| \leq \epsilon$. This means that $\{f_n\}_{n\geq 0}$ converges uniformly to f. □

Remark. The theorem above holds more generally when we have a complete metric space on the right and A is any topological space.

Theorem 11. *Uniform limit of continuous functions is continuous.*

Proof. Let $A \subseteq \mathbb{R}^m$ and $\{f_n\}_{n\geq 0}$ be a sequence of continuous functions on A converging uniformly to some f. Given $x, y \in A$, we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

for every n. We control the first and last terms by uniform convergence and the middle term by continuity of f_n, therefore given $\epsilon > 0$, we can choose N, δ such that for $n \geq N, |x - y| \leq \delta$, all three terms are less than $\epsilon/3$. Continuity of f follows. As remarked earlier, the same proof applies in more general situations. □

While uniform limit of continuous functions is continuous, the same cannot be said about differentiability. A common example is the Weierstrass function

$$f(x) = \sum_{k\geq 0} 2^{-k} \cos(3^k x)$$

which is the uniform limit of the partial sums which are differentiable, but f is differentiable nowhere, but continuous everywhere. The problem is because while the functions converge, their derivatives need not.

Theorem 12. *(Limit and derivative) Let $J \subseteq \mathbb{R}$ be a bounded interval and let $\{f_n\}_{n\geq 0}$ be a sequence of functions $J \to \mathbb{R}$. Suppose $\exists x_o \in J$ such that $\{f_n(x_0)\}_{n\geq 0}$ converges and suppose that the sequence $\{f'_n\}_{n\geq 0}$ exists on J and converges uniformly to a function g. Then $\{f_n\}_{n\geq 0}$ converges uniformly on J to a function f with $f' = g$.*

Proof. Let $a < b$ be the ends of J and $x \in J$. Then we have by the mean value theorem, for every m, n some y between x, x_0 such that

$$\begin{aligned} f_m(x) - f_n(x) &= f_m(x_0) - f_n(x_0) - (x - x_0)(f'_m(y) - f'_n(y)) \\ \implies \|f_m - f_n\|_J &\leq |f_m(x_0) - f_n(x_0)| + (b - a)\|f'_m - f'_n\|_J \end{aligned}$$

We control the first term by the convergence of $f_n(x_0)$ and the second term by the uniform convergence of f'_n. It follows that $\{f_n\}_{n\geq 0}$ satisfies the Cauchy criterion, hence uniformly converges to some function f. Next we compute teh derivative of f at some $c \in J$. Given $x \in J$, there is a z between x, c such that

$$(f_m(x) - f_n(x)) - (f_m(c) - f_n(c)) = (x - c)(f'_m(z) - f'_n(z))$$
$$\implies \left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \|f'_m - f'_n\|_J$$

for $x \neq c$. Given $\epsilon > 0$, find N so that for $m, n \geq N$, the right side is bounded by ϵ. Then taking the limit as $m \to \infty$, the inequality remains and because $|\cdot|$ is continuous, we have

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \epsilon \text{ for } x \neq c, n \geq N.$$

Obtain $\delta > 0$ so that for $0 < |x - c| < \delta$, the send term is within ϵ of $f'_n(c)$. Next increasing N if necessary, make sure that $\|f'_n - g\| \leq \epsilon$. Together, we get by the triangle inequality, for $0 < |x - c| < \delta, n \geq N$

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq 3\epsilon.$$

Therefore, f is differentiable at c, and since c was arbitrary, f is differentiable with derivative g. □

Theorem 13. *(Limit and integral) Let $\{f_n\}_{n\geq 0}$ be a sequence of integrable functions converging uniformly to f on $[a, b]$, then f is integrable with $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.*

Proof. First, given $\epsilon > 0$, choose N such that for $m, n \geq N$, we have $|f_m(x) - f_n(x)| < \epsilon$, then by monotonicity of integrals, we have $|\int_a^b f_m - \int_a^b f_n| \leq \epsilon(b - a)$, hence the integrals converge. Next, let $\mathcal{P}$ be any partition of $[a, b]$, given $\epsilon > 0$ choose N so that for $n \geq N$, we have $\|f_n - f\| < \epsilon$, then

$$\begin{gathered} f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon \\ \implies U(f; \mathcal{P}) \leq U(f_n; \mathcal{P}) + \epsilon(b - a) \\ \text{and } L(f; \mathcal{P}) \geq L(f_n; \mathcal{P}) - \epsilon(b - a) \end{gathered}$$

Since $\{f_n\}_{n\geq 0}$ are all integrable, it follows that f is integrable. Morover, from the same inequalities above we get for $n \geq N$, $|\int_a^b f - \int_a^b f| \leq \epsilon(b - a)$, therefore $\int_a^b f$ is the limit of $\int_a^b f_n$. □

Remark. Here to we have more general theorems called Monotone Convergence Theorem and Dominated Convergence Theorem. Moreover, if we have a function over a curve in $\mathbb{C}$, that is a uniform limit of functions, then the integral and limits interchange using parametrization of the curve and the fact that the real and imaginary parts are uniformly convergent.

Theorem 14. *(Dini's Theorem) Suppose $\{f_n\}_{n\geq 0}$ is monotone sequence of continuous functions on $I = [a, b]$ converging pointwise to a continuous function f. Then the convergence is uniform.*

Proof. We will assume that $\{f_n\}_{n\geq 0}$ is increasing, the other case is similarly proved. First, fix an $x \in I, \epsilon > 0$. Pick N such that $f(x) - f_N(x) < \epsilon$. Then the same holds for any $n \geq N$ because of monotonicity. Then for any $y \in I$, we have

$$|f(y) - f_n(y)| \leq |f(y) - f(x)| + |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)|.$$

By continuity of f, f_N, we can control the first and third terms. By choice of N, we control the second term. Suppose $\delta > 0$ is chosen so that for $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$ and $|f_N(x) - f_N(y)| < \epsilon$. Then by monotonicity, for $n \geq N$, we have

$$f_N(x) - \epsilon < f_N(y) \leq f_n(y) \leq f(y) < f(x) + \epsilon.$$

Therefore, $|f_N(y) - f(y)| < 3\epsilon$. Thus, $|f(y) - f_n(y)| \leq 6\epsilon$ for $|x - y| < \delta, n \geq N$.

Thus, for every x, we have a $\delta_x > 0, N_x$ where there are inequalities as above. We cover I by open intervals of radii δ_x and by compactness, choose a finite subcover, say intervals $B(x_1, \delta_{x_1}), \ldots, B(x_k, \delta_{x_k})$ and take $N = \max\{N_{x_1}, \ldots, N_{x_k}\}$. Then for any $y, n \geq N$ we have $|f(y) - f_n(y)| \leq 6\epsilon$ because y is within δ_{x_i} of some x_i. Therefore, $\{f_n\}_{n\geq 0}$ converges uniformly to f. □

Remark. Note that above, we will have to use half intervals at the end point, but that is not a problem because we take open sets instead of open intervals. The exact same proof holds when I is replaced by arbitrary compact spaces.

5 Series of functions

Definition 2. *If $\{f_n\}_{n\geq 0}$ is a sequence of functions on some domain D with values in R, we define the partial sums $s_n = f_1 + \cdots + f_n$. If the sequence $\{s_n\}_{n\geq 0}$ converges (pointwise or uniformly) on D to a function f, we say $\sum f_n$ converges to f on D. If $\sum |f_n|$ converges, we say $\sum f_n$ is absolutely convergent. If $\{s_n\}_{n\geq 0}$ converges uniformly, then we say $\sum f_n$ is uniformly convergent.*

Theorem 15. *(Cauchy criterion) Let $\{f_n\}_{n\geq 0}$ be a sequence of functions on D, then $\sum f_n$ is uniformly convergent if and only if the sequence of partial sums satisfy the Cauchy criterion.*

Theorem 16. *(Weierstrass M-test) Let $\{M_n\}_{n\geq 0}$ be a sequence of positive real numbers such that $\|f_n\| \leq M_n$. If $\{M_n\}_{n\geq 0}$ converges, then $\sum f_n$ is uniformly convergent.*

Proof. We have $|f_n(x) + \cdots + f_m(x)| \leq M_n + \cdots + M_m$ for $n \leq m$ and $x \in D$. Since $\sum M_n$ converges, it is Cauchy, therefore, by Cauchy criterion, $\sum f_n$ converges uniformly. Note that this applies to complex valued functions as well. □

Definition 3. *A series of functions (real of complex valued) $\sum f_n$ is said to be a power series around $x = c$ if for every n, $f_n = a_n(x-c)^n$ for some $a_n \in \mathbb{C}$ in a neighbourhood of c.*

Let us look at where $\sum a_n x^n$ converges (taking $c = 0$ for convenience). Recall the root test, $\sum a_n x^n$ converges if $\limsup_{n\to\infty} |a_n x^n|^{1/n} < 1$ and diverges if > 1. Define

$$R = 1/\limsup_{n\to\infty} |a_n|^{1/n}$$

with the convention $1/\infty = 0, 1/0 = \infty$.

Then, if $|x| < R$, the series converges absolutely, if $|x| > R$ it diverges and we cannot say what happens when $|x| = R \neq 0$. So, there is an open disc region of convergence and R is called the radius of convergence. We can obtain the radius of convergence by the ratio test, provided the coefficients are nonzero.

Theorem 17. *(Cauchy-Hadamard theorem) If R is the radius of convergence, then the power series $\sum a_n x^n$ converges absolutely if $|x| < R$ and diverges if $|x| > R$.*

Theorem 18. *Let R be the radius of convergence of $\sum a_n x^n$, and let $K \subset B(0, R)$ be a compact subset, then the series converges uniformly on K.*

Proof. Since K is bounded, there is some $0 \leq c < 1$ such that $|x| < cR \, \forall \, x \in K$. Then

$$|\sum_{n\geq m} a_n x^n| \leq \sum_{n\geq m} |a_n|(cR)^n \, \forall \, x \in K.$$

Since $c < 1$¡ the right side converges and we can make the bound independent of $x \in K$, therefore the series converges uniformly on K. □

Corollary 2. *$\sum a_n x^n$ is continuous on $B(0, R)$ and can be integrated term by term on any compact set.*

Proof. By uniform convergence on compact sets, integration can be done term by term and since around each $x \in B(0, R)$ we can find a compact set, the series is continuous at each x. □

Theorem 19. *(Differentiation theorem) A power series can be differentiated term by term in the region of convergence. In fact, if $f(x) = \sum_{n\geq 0} a_n x^n$, then $f'(x) = \sum_{n\geq 1} n a_n x^{n-1}$ for $|x| < R$ and both series have the same radius of convergence.*

Proof. Since $\lim_{n\to\infty} n^{1/n} = 1 = \limsup n^{1/n}$, $\limsup |na_n|^{1/n} \leq \limsup |a_n|^{1/n}$. On the other hand, since $n^{1/n} > 1$, we have $\limsup |a_n|^{1/n} \leq \limsup |na_n|^{1/n}$. Therefore, the two series have the same radius of convergence.

On any compact subset, both series converge uniformly, and each partial sum's derivatives also converge to the f' above. Therefore, f is differentiable with the given derivative.

We give another proof. Let $z \in B(0, R)$ and choose $\delta > 0$ such that $|z| + \delta < R$. For $|h| < \delta$,

$$f(z + h) = \sum a_n(z^n + hz^{n-1} + h^2 P_n(z, h)$$

where $P_n(z,h) = \sum_{k\geq 2} \binom{n}{k} h^{k-2} z^{n-k}$. Observe that

$$|P_n(z, h)| \leq \sum_{k\geq 2} \binom{n}{k} \delta^{k-2} |z|^{n-k} = P_n(|z|, \delta).$$

Since everything is uniformly convergent (we are restricting to a compact subset around z), we can rearrange the terms freely, and we have

$$f(z + h) - f(z) - \sum_{n\geq 1} na_n z^{n-1} h = h^2 \sum_{n\geq 2} a_n P_n(z, h).$$

Since the series on the left converges, so does the series on the right. We get

$$\left| \frac{f(z + h) - f(z)}{h} - \sum_{n\geq 1} na_n z^{n-1} \right| = |h \sum_{n\geq 2} a_n P_n(z, h)| \leq h \sum_{n\geq 2} |a_n| P_n(|z|, \delta).$$

The right side of this inequality is hc where c is a finite constant independent of h. Therefore, f is differentiable with the derivative f' as above. □

Lemma 7. $\lim_{n\to\infty} n^{1/n} = 1$

Proof. Write $n^{1/n} = (1 + \alpha)$ for $\alpha > 0$, then

$$\begin{aligned} n = (1 + \alpha)^n &= 1 + n\alpha + \frac{n(n - 1)}{2}\alpha^2 + \sum_{k\geq 2} \binom{n}{k} \alpha^k \\ &\geq 1 + \frac{n(n - 1)}{2}\alpha^2 \end{aligned}$$

Thus, $\alpha \leq \sqrt{2/n}$, hence $\alpha \to 0$ in the limit. Therefore, $n^{1/n} \to 1$. □

Theorem 20. *(Abel's theorem) Let $\sum_{n\geq 0} a_n z^n$ be a power series with radius of convergence $R \geq 1$. Assume $\sum a_n$ converges and let $0 \leq x < 1$. Then*

$$\lim_{x\to 1} \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} a_n.$$

Proof. For any N, x we have

$$\left| \sum_{n\geq 0} a_n - \sum_{n\geq 0} a_n x^n \right| \leq \left| \sum_{n\geq 0} a_n - \sum_{n=0}^{N} a_n \right| + \left| \sum_{n=0}^{N} a_n - \sum_{n=0}^{N} a_n x^n \right| + \left| \sum_{n=0}^{N} a_n x^n - \sum_{n\geq 0} a_n x^n \right|.$$

We choose N so as to control the first and last terms, and such N exists because both series are convergent, i.e., $\sum a_n x^n$ converges absolutely on $B(0, R)$ and $\sum a_n$ converges. Using continuity of polynomials (both real and complex valued), we can choose x to be close to 1 (having first chosen N) so that the middle term is also bounded. This way, we can bound the left side by any $\epsilon > 0$ and the theorem follows. □

6 Algebra of power series

First note that if $\sum a_n$ converges to a, and $\sum b_n$ to b, then $\sum(a_n + b_n)$ converges to $a + b$. Given two series f, g we define their addition, multiplication and scalar multiplcation in a formal manner, by changing the coefficients correspondingly.

Theorem 21. *If f, g are power series converging absolutely on the disc $B(0, r)$, then $f + g, fg$ also converge on this disc. If $\alpha \in \mathbb{C}$, then αf also converges absolutely on the disc. Moreover, $(f + g)(z) = f(z) + g(z), (fg)(z) = f(z)g(z), (\alpha f)(z) = \alpha f(z)$.*

Proof. We give a proof for the product. Write formally, $f = \sum a_n T^n, g = \sum b_n T^n$, then $fg = \sum c_n T^n$ where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. For $0 < s < r$, there is an N such that for $n \geq N, |a_n|, |b_n| \leq 1/s^n$. Therefore, for $n \geq N$

$$|c_n| \leq \frac{n+1}{s^n} \implies \limsup_{n\to\infty} |c_n|^{1/n} \leq \frac{1}{s}$$

where we have used $\lim(n + 1)^{1/n} = 1$. Since $s < r$ was arbitrary, we have $\limsup |c_n|^{1/n} \leq 1/r$ and the formal product defined above converges on the disc (and potentially beyond the disc).

Let f_N, g_N denote the Nth partial sums of f, g respectively. For any $z \in B(0, r), f_N(z), g_N(z)$ converge to $f(z), g(z)$ respectively. We look at $fg(z) - f_N(z)g_N(z)$ The coefficients of $fg, f_N g_N$ match upto T^N, so we have

$$|(fg)(z) - f_N(z)g_N(z)| \leq \sum_{n=N+1}^{\infty} \sum_{k=0}^{n} |a_k b_n - k||z|^n.$$

On the right side, we have the tail of fg beyond z^N. Since $fg(z)$ converges, the right side can be made as small as possible. Therefore, $f_N(z)g_N(z)$ converges to $fg(z)$. Hence, $(fg)(z) = f(z)g(z)$.

For scalar multiplication, we have $\limsup |\alpha|^{1/n}|a_n|^{1/n} = \limsup |a_n|^{1/n}$, so the radius of convergence is the same and it is easy to show that $(\alpha f) = \alpha(f)$.

Similarly, for addition, we have the same or larger radius of convergence, and the convergence can be established because the limit of finite sums is the finite sum of limits. □

If $f(z) = \sum_{n\geq 0} a_n z^n$ and if $a_0 \neq 0$, then we can construct a formal inverse. Take $f^{-1} = \sum_{n\geq 0} b_n z^n$, then we want $a_0 b_0 = 1$ and for each $n \geq 1$,

$$a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n = 0.$$

Since $a_0 \neq 0$, we can recursively solve for b_n.

Theorem 22. *(Inverse of power series) Suppose f has a non zero radius of convergence and non zero constant term. Let g be the formal inverse of f, then g also has a non zero radius of convergence.*

Proof. Multiplying by a constant (we have seen that this is a well behaved operation) if necessary, we may assume f has constant term 1. Write $f = 1 + a_1 T + a_2 T^2 + \ldots$ and let the formal inverse be $g = 1 + b_1 T + b_2 T^2 + \ldots$. Because f has a positive radius of convergence, there is an $\infty > A > 0$ such that for every $n \geq 0, |a_n| \leq A^n$. For example, there is an N such that for $n \geq N$, we have $1/r \leq \sup_{k\geq n} |a_n|^{1/n} \leq 1/r + 1 < \infty$ where r is the radius of convergence. This means that every for $n \geq N$, we have $|a_n| \leq (1/r + 1)^N$. We may choose A sufficiently larger than $(1/r + 1)$ so as to bound the other $|a_n|, n < N$.

Now, we claim that for $n \geq 1, |b_n| \leq 2^{n-1} A^n$. First, $b_1 = -a_1$, therfore $|b_1| \leq 2^{1-1} A^1$. Now assume that $|b_k| \leq 2^{k-1} A^k$ for $k < n$. We know

$$b_n = -a_n - a_{n-1} b_1 - a_{n-2} b_2 - \cdots - a_1 b_{n-1}$$

hence

$$|b_n| \leq A^n + A^n + 2A^n + 4A^n + \cdots + 2^{n-2} A^n = 2^{n-1} A^n.$$

Thus, by induction, for each $n \geq 1, |b_n| \leq 2^{n-1} A^n$, hence $\limsup |b_n|^{1/n} \leq 2A < \infty$. Thus the formal inverse also has a non zero radius of convergence. □

Theorem 23. *(Composition of power series) Let* $f(z) = \sum_{n\geq 0} a_n z^n, h(z) = \sum_{n\geq 1} b_n z^n$ *be convergent power series with* h *having zero constant term. Assume* f *is absolutely convergent on* $|z| \leq r$ *with* $r > 0$ *and that there is an* $s > 0$ *such that* $\sum_{n\geq 1} |b_n| s^n \leq r$*. Let* $g = f \circ h$ *be the formal power series obtained by* $g(T) = \sum_{n\geq 0} a_n (\sum_{k\geq 0} b_k T^k)^n$*. Then* g *converges absolutely for* $|z| \leq s$ *and for such* z*, we have* $g(z) = f(h(z))$*.*

Proof. Formally we have the composition of two power series $f = \sum_{n\geq 0} a_n T^n, h = \sum_{n\geq 0} b_n T^n$ given by $f \circ h = \sum_{n\geq 0} c_n T^n$ where $c_n = \sum_{k\geq 0} a_n b_k(n)$ with $b_k(n)$ being the coefficient of T^n in h^k given by $\sum b_{i_1} \ldots b_{i_k}$ where the indices come from the set $\{(i_1, \ldots, i_k) | 0 \leq i_j, \sum_{j=1}^k i_j = n\}$. The problem is that each c_i being an infinite sum, it need not be defined. However, with the restriction that $b_0 = 0$, the power h^k cannot contain T^n for $k > n$, hence each c_n becomes a finite sum and hence is well defined.

For $|z| \leq s, |h(z)| \leq |h(s)|$, next we know that $\sum_{n\geq 1} |b_n| s^n \leq r$, therefore for any n, we have

$$|b_1| s + \cdots + |b_n| s^n \leq r.$$

Taking the kth power, the absolute value of the coefficient of s^n is less than r^k/s^n. Notice that this is also the absolute value of the coefficient of z^n in h^k, which we call h_n^k.

Let the coefficient of $z^n, n > 0$ in the formal composition be denoted g_n, then

$$|g_n| \leq \sum_{k=1}^{n} |a_k| \frac{r^k}{s^n} \leq \frac{1}{s^n} \sum_{k\geq 0} |a_k| r^k.$$

Because f is convergent on $|z| \leq r$, the right side is c/s^n for some constant c, therefore,

$$\limsup |g_n|^{1/n} \leq 1/s,$$

hence g has radius of convergence greater than s, therefore the formal composition is defined on $|z| \leq s$.

Next we show that the power series of the formal composition evaluates to the composition of f, h as functions. This composition of functions is well defined because anything in the disc of radius s lands inside a disc of radius r under h where f is convergent. Let f_n denote the nth partial sum of f.

Observe that the first n coefficients of the polynomial $f_n \circ h$ agree with the first n coefficients of g and when writing the coefficients of g as a sum, the remaining coefficients of $f_n \circ h$ agree with partial sums. Therefore, the power series $f_n \circ h - g$ is bounded by the nth tail of $\tilde{g} = \sum_{n\geq 0} |g_n| z^n$, which we write g_n^*, then

$$|g - f \circ h| \leq |g - f_n \circ h| + |f_n \circ h - f \circ h| \leq g_n^* + |f_n \circ h - f \circ h|.$$

Now, since $\tilde{g}$ is absolutely convergent, we can bound the tail g_n^* by ϵ for sufficiently large n. By uniform convergence of f_n to f on the closed disc of radius r and the fact that h lands inside this disc, we can bound the second term by ϵ for sufficiently large n. Together, we conclude that $g(z) = f(h(z))$ for $|z| \leq s$ as required. $\square$

Remark. For results on composition of general power series, see [3].

7 Analytic functions

Now we consider functions $f: D \to \mathbb{C}$ where D is an open or compact subset of $\mathbb{C}$. If f is a function defined on a neighbourhood of $z_0 \in \mathbb{C}$, then f is said to be analytic at z_0 if there is a series $\sum_{n\geq 0} a_n (z - z_0)^n, a_n \in \mathbb{C}$ and a radius $r > 0$ such that this series converges on $|z - z_0| < r$ and for such z, it converges to $f(z)$. Note that we require f to be defined on an open set so that we have open balls around z_0.

By shrinking the radius if necessary, it is easy to see that sums, products and quotients of functions analytic at z_0 are also analytic at z_0. Now suppose $g: U \to V, f: V \to \mathbb{C}$ are continuous functions where U, V are open and that g, f are analytic at $z_0, g(z_0) = w_0$ respectively. By continuity of g, we may suppose there is a ball U_1 around z_0 of radius r where g has a series expansion, and a ball U_2 of radius s around w_0 where f has a series expansion such that $g(U_1) \subset V_1$.

Suppose that $g = w_0 + \sum_{k\geq 1} b_k(z-z_0)^k, f = \sum_{n\geq 0} a_n(z-w_0)^n$ on U_1, V_1 respectively. Then, their composition is defined and analytic, because while composing we are going to take $g - w_0$ which has zero constant term.

Theorem 24. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a power series with radius of convergence r, then f is analytic on $D(0, r)$.*

Proof. Let $z_0 \in D(0, r)$ and let $s > 0$ be such that $D(z_0, s) \subseteq D(0, r)$. For $z \in D(z_0, s)$, write $z = z_0 + (z - z_0)$, then $f(z) = \sum_{n\geq 0} a_n(z_0 + (z - z_0))^n$. This is a double summation and we seek to interchange the order of summation. The finite sums of absolute values of the terms involved are bounded by $\sum_{n=0}^{m} a_n(|z_0| + |z - z_0|)^n$ for some m which have finite sumpremum because it is the value of f evaluated at $|z_0| + |z - z_0|$ which is a positive real number in $D(0, r)$ where f converges absolutely. Therefore, we may apply Fubini-Tonelli theorem for summation and interchange the order of summation to get $f(z) = \sum_{n\geq 0} b_n(z - z_0)^n$ for some $b_n \in \mathbb{C}$. □

Theorem 25. *(Zeroes of an analytic function) Suppose $f : U \to \mathbb{C}$ is an analytic function where U is some connected open subset of the complex plane. Then the zeroes of f are isolated or f is identically zero.*

Proof. Let z_0 be a zero of f and V a ball around z_0 where f has an expansion $\sum_{n\geq 0} a_n(z - z_0)^n$. First suppose f is not identically zero on V, then some $a_n \neq 0$, say a_m is the smallest m for which $a_m \neq 0$, so that $f = (z - z_0)^m \sum_{n\geq m} a_m(z - z_0)^{n-m}$ with $a_m \neq 0$. Then clearly, z_0 is the only root of f in V, thus z_0 is an isolated root of f.

Next, suppose f is identically zero on V. Let C_0 be the component of z_0 in the subset of zeros $Z(f)$ of f. Note that C_0 is closed because $Z(f)$ is closed by the continuity of f (and components are closed). Now, suppose $z_1 \in \partial C_0$, then z_1 is a limit point of C_0 as C_0 is connected and closed. Expanding f as a power series around zero shows that f must be identically zero in a neighbourhood of z_1 as z_1 is not isolated. This would mean that an open neighbourhood around z_1 is part of C_0 contradicting the fact that $z_0 \in \partial C_0$, therefore $\partial C_0 = \emptyset$. This means that C_0 is a non empty clopen subset of U, hence $C_0 = U$ because U is connected. Thus, f is identically zero on U. □

Remark. The same proof applies to real analytic functions. If f is a nonzero analytic function, then it can its zeroes are isolated. This means that in every compact set there are finitely many zeroes, hence the zeroes are nowhere dense, and countable. The above theorem goes by the name Identity Theorem.

7.1 Binomial power series

Given $\alpha \in \mathbb{C}$, look at the power series

$$\sum_{n\geq 0} \binom{\alpha}{n} z^n$$

where $\binom{\alpha}{n} = \alpha(\alpha - 1) \ldots (\alpha - n + 1)/n!$ is the generalized binomal coefficient. We apply the ratio test (taking α to not be an integer, otherwise the series is finite),

$$\limsup_{n\to\infty} \left| \binom{\alpha}{n+1} / \binom{\alpha}{n} \right| = \limsup_{n\to\infty} \left| \frac{\alpha - n}{n+1} \right| = 1$$

therefore, the series converges for $|z| < 1$.

A note on the binomial series. In general, because $\mathbb{C}$ is algebraically closed, it has a lot of roots. Here's a flow of ideas that leads to the above expansion. One first defines the integer powers of real numbers, then using the completeness properties of $\mathbb{R}$, one can define rational powers. Then, using rational approximations of irrationals, we can define irrational powers of reals. Next, we prove certain smoothness properties of the functions x^a, a^x where a is some fixed nonnegative number (we can take a to be negative for x^a).

One then proves that x^a is real analytic, i.e., the remainder term in its Taylor expansion goes to zero, this way we get a Taylor series of $(1+x)^a$ which is a power series converging for $|x| < 1, x \in \mathbb{R}$. We can then extend this power series to $\mathbb{C}$, i.e., making $x, a \in \mathbb{C}$, noting that the radius of convergence doesn't change because ultimately it is a calculation in $\mathbb{R}$.

To answer whether this is really $(1+z)^\alpha$, we need to develop a theory of exponentials over the complex plane. What one does is to extend the power series of $e^x, \ln(1+x)$ to $\mathbb{C}$ in appropriate domains. Then to compute $(1+z)^\alpha$, we first take the logarithm of $(1+z)$, multiply it by α, and then exponentiate the result. One can see that doing this gives the above power series in the limit.

7.2 Inverse and open mapping theorem

Let U be an open subset of $\mathbb{C}$ and $f\colon U \to V$ analytic. We say that f is an analytic isomorphism if V is open and there is a $g\colon V \to U$ analytic such that $f \circ g = id_V, g \circ f = id_U$. We say that f is a localy analytic isomorphism at $z_0 \in U$ if there is an open neighbourhood $U_0 \subseteq U$ of z_0 such that $f|_{U_0}$ is an analytic isomorphism.

Theorem 26. *1. let $f(T) = \sum_{n\geq 0} a_n T^n$ be a formal power series with $a_0 = 0, a_1 \neq 0$, then there is a unique formal power series $g(T)$, with constant term zero, such that the formal power series $f \circ g = T$.*

2. If f is convergent, then so is g, where f, g are as above.

3. Let f be an analytic function on an open set U containing a point z_0. Suppose $f'(z_0) \neq 0$, then f is a local analytic isomorphism at z_0.

Proof. 1. Write $f(T) = a_1 T + \sum_{n\geq 2} a_n T^n$ and take $g(T) = \sum_{n\geq 1} b_n T^n$ arbitrary. We solve for the coefficients b_i using the relation $f \circ g = T$, then we will have equations of the type

$$a_1 b_n - P_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}) = 0$$

with $a_1 b_1 = 1$ and P_n are some polynomials. We can now solve for the b_i recursively, because $a_1 \neq 0$. Since the same relations must hold for any g with constant term zero, such a g is unique.

2. We may take $a_1 = 1$ because it doesn't change the radius of convergnece of the formal inverse. Take $f^* = T + \sum_{n\geq 2} a_n^* T^n$ where a_n^* are real numbers such that $|a_n| \leq a_n^*$. Let $\phi(T) = \sum_{n\geq 1} c_n T^n$ be the formal inverse of f^*. We then have

$$c_n - P_n(a_2^*, \ldots, a_n^*, c_1, \ldots, c_{n-1}) = 0.$$

It is easy to see that P_n is a polynomial with positive coefficients and $c_1 = 1 > 0$, therefore, by induction, each c_n is nonnegative. Moreover, since b_n satisfies a similar equation, by induction we also have $|b_n| \leq c_n$ for $|a_n| \leq a_n^*$. Thus, we choose a_n^* so that the c_n have a positive radius of convergence.

Choose $A > 0$ such that $|a_n| \leq A^{n-1}$ for every $n \geq 2$, and take $a_n^* = A^{n-1}$. Such an A exists because the series $\sum_{n\geq 2} a_n T^n$ also has a positive radius of convergence. Then formally,

$$f^*(T) = T + \sum_{n\geq 2} A^{n-1} T^n = T + \frac{AT^2}{1 - AT}.$$

We put $T = \phi(T)$ and solve formally to get

$$\phi(T) = \frac{T}{1 + AT} = T(1 - AT + (AT)^2 - \ldots).$$

It is clear that ϕ has a positive radius of convergence, therefore g also has a positive radius of convergence.

3. Let f be analytic on a neighbourhood of z_0, say $f(z) = a_0 + a_1(z - z_0) + \dots$. Then, we can differentiate (which is a local operation) f term by term, so $a_1 = f'(z_0)$. By translation (which retains local analytic isomorphisms), we may take $z_0 = a_0 = 0$, then given $f'(0) \neq 0$, there is a neighbourhood U around 0 where f is injective. We can take U to be an open disc centred at the origin and let g be the inverse of f. Take an open disc V around the origin such that $g(V) \subseteq U$ (such a V exists by continuity of g), then $f(U) \supseteq f(g(V)) = V$.

   Using continuity of f, take $U_0 = f^{-1}(V) \cap U$ open, then $f\colon U_0 \to V$ is a bijection with f and its inverse g being analytic, therefore f is a local analytic isomorphism.

$\square$

There are many more results about power series, but these require tools from Complex Analysis.

References

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